Suppressing nonrevisiting paths

Frédéric BOSIO

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Abstract

In this article, we give, under some hypothesis, a couterexample to the nonrevisiting path conjecture and this also might refute conjectures that are known to be equivalent to it, especially Hirsch's conjecture.

Introduction

In 1957, W.M. Hirsch asked if every d-dimensional polytope with n facets has diameter at most $n - d$. This is now referred as Hirsch conjecture. The original question was stated also for unbounded polyhedra, but in this case, we know it is false [K-W]. Many works have been done in order to prove or refute it and it has been established for some special classes of polytopes ([Na],[Kr],[Kl]). We can also look at [Zi] for a survey of this theory.

We know Hirsch conjecture equivalent to the two following conjectures [K-W]:
i) The nonrevisiting path conjecture, which states that given two vertices of a polytope, there always exists a path joining them whose intersection with any facet is connected.
ii) The d-step conjecture: Consider a d-dimensional simple polytope with $2d$ facets and two vertices x and y lying on complementary sets of facets. Then there is a path of length d from x to y.

Notice this other formulation of this conjecture: Consider two $(d - 1)$-simplices in $\mathbb{R}^{d-1}$ containing a common point p. Then there is a numbering $A_1, ..., A_d$ and $A'_1, ..., A'_d$ of their respective vertices such that for any i, p belongs to the convex hull of $A_1, ..., A_i, A'_{i+1}, ..., A'_d$.

On the other hand, many polytopes have been found that are sharp (i.e. meet the bound $n - d$) [H-K], [F-H], which casts doubt on the validity of the conjecture.

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We show here that, under some hypothesis about combinatorial flips of polytopes, it is possible, using a technique that we call path-flipping, to eliminate nonrevisiting paths, and that leads to a refutation of the nonrevisiting path conjecture.

1   Notations and recalls

We present here some basic definitions, notations and technics about simple polytopes.

A polytope means the convex hull of a finite (nonzero) number of points in a real affine space. It also can be seen (and preferably for us) as the intersection of some closed half-spaces of an affine space, that is bounded and with nonempty interior.

Here do we only consider simple polytopes (the number of facets containing a given vertex is equal to the dimension of the polytope). Also, we are only interested in combinatorial polytopes, i.e. isomorphism types of face posets of polytopes. So we will call polytope a combinatorial polytope.

NOTATION **1** *The dimesion of a polytope P will be noted d_P, its number of facets n_P and its number of vertices n'_P.*

We basically indicate vertices of a simple polytope by the set of facets that contain it, for example the vertex of a 3-dimensonal polytope lying on the facets F,F' and F'' ought to be noted $(FF'F'')$.

DEFINITION **1.1** *The two facets that contain an extremity of an edge of a simple polytope an not the other one are called the* extremal facets *of this edge.*

Notice that a facet containing a vertex of a simple polytope is the extremal facet of exactly one edge containing this vertex.

DEFINITION **1.2** *Let P be a polytope. A* path *in P is a finite sequence $(v_0, ..., v_l)$ of vertices of P such that for any $i, 1 \le i \le l$, v_{i-1} and v_i are adjacent (i.e. are joined by an edge) in P. We also demand that $v_i \ne v_j$ if $i \ne j$.*

The vertex v_0 is called the origin *of the path, the vertex v_l its* end *and the integer l its* length.

For $1 \le i \le l$, the facet of P (supposed simple) that contains v_i and not v_{i-1} is called the ith arrival facet of the path. The facet that contains v_0 and not v_1 is called its start facet.

If v and v' are vertices of a polytope P, their distance is the minimal length of a path whose origin is v and whose end is v' (such a path always exist) and we call diameter of P the maximal distance between two of its vertices.

DEFINITION **1.3** *A path* $(v_0, ..., v_l)$ *in a simple polytope* P *is called* nonrevisiting *if, for any facet* F *of* P*, the set of integers* i *for whose* v_i *belongs to* P *is an interval of* $\mathbb{N}$.

REMARK **1.1** *A path is nonrevisiting if and only if no two of its arrival facets are equal and none of them contains its origin.*

We now present some manipulations on the simple polytopes. We can refer to [F-H] for blending and wedging, or [Ti] for flips.

1.1 Blendings

Consider two simple polytopes P and Q of the same dimension, a vertex v'_P (resp. v'_Q) of P (resp. of Q), and a one-to-one correspondance ϕ between the facets of P containing v'_P and the facets of Q containing v'_Q. We can then construct a simple polytope $P\#_\phi Q$, or simply $P\#Q$ when no confusion is possible, called the blending of P and Q (at v'_P and v'_Q relatively to ϕ) by cutting off small neigbourhoods of the vertices and glueing their complements, identifing a facet with its image under ϕ (when it has one).

DEFINITION **1.4** *A face of* P *will be called* stable *if it doesn't contain* v'_P *and* unstable *if it does.*

In the sequel, we won't distinguish stable faces from their natural image under the blending.

PROPOSITION **1.2** *We have* $n_{P\#Q} = n_P + n_Q - d$ *and* $n'_{P\#Q} = n'_P + n'_Q - 1$.

DEFINITION **1.5** *Consider an edge* e *in* P *whose vertices are* v'_P *and another, noted* v*. Then there is exactly one edge* (v, v') *in* $P\#Q$ *which contains* v *and crosses the glueing locus. The extremal facet of* (v, v') *in* $P\#Q$ *which contains* v' *is called the* virtual extremal facet *of the edge* e*.*

Consider now a path in P *whose origin is* v'_P*. Then the virtual extremal facet of its first edge is called the* virtual start facet *of this path, and the vertex* v' *thereup is called the* virtual origin *of this path.*

Though the construction of a blending is symmetric in P and Q, our constructions won't be. We will call P the left polytope and Q the right one, as well as we call things related to P (vertices, stable facets, ...) the left ones and those related to Q the right ones.

1.2 Flips

Consider a simple polytope P of dimension at least 3 and a maximal simplicial face S of P (maximal meaning that no greater simplicial face contains it), that is neither P nor a facet of P. Then we call combinatorial flip of P along S the following transformation :

Let $F_1, ..., F_k$ the facets of P whose intersection is S and $G_1, ..., G_l$ the others facets that meet S. Let P_S the poset obtained by keeping the same set of facets as P, removing the vertices of S and introducing the vertices that lie on every G_i and all but one F_j. We then take as face of P_S any intersection of facets that contains a vertex. Remark that the vertices that we have introduced are pairwise adjacent. We say that P_S is obtained from P by a (combinatorial) flip along S.

With our definition, P and P_S have naturally the same set of facets.

N.B. Generally, authors consider that the operations of truncation of a vertex (remark such an operation introduces a new facet) or collapsing a simplicial facet onto a vertex (if P is not a simplex) also are flips, and sometimes even the operations of appearance and disappearance of a simplex. We do not.

DEFINITION **1.6** *Consider a polytope P and a polytope Q obtained from P by a flip along a face E. Consider a face F of P. Its* strict transform *in Q is the intersection (in Q) of the facets that contain it (in P).*

Besides, we call new *a face (and especially a vertex) that is in Q and not in P (i.e. the intersection in P of the facets containing this face is empty in P).*

Note that the strict transform of a face F of P is empty if and only if F lies in S.

REMARK **1.3** *Let Q a polytope obtained from a polytope P by a flip. Then the new facets of Q form a maximal simplicial facet S' and the flip of Q along S' gives back P.*

DEFINITION **1.7** *An edge (or a simplicial face) of a simple polytope is called* (combinatorially) flippable *if the combinatorial flip of the polytope along this face can be performed an gives rise to another polytope.*

The hypothesis that will allow us to construct our counterexample is precisely related to flippability of edges.

Principal Hypothesis: *An edge of a ≥ 3-dimensional simple polytope is flippable if and only if its two extremal facets are disjoint.*

The only if part is clear since else the strict transforms of these facets would have a nonconnected intersection in the new polytope, the if part is unknown (as far as I

know). However, there are many cases in which we know it is true. Let's give two representative examples:

PROPOSITION **1.4** *The hypothesis is true for edges resulting of a blending (i.e. edges that cross the glueing locus).*

PROOF Consider a polytope P with a vertex v'_P, a polytope Q with a vertex v'_Q and a bleding $P\#Q$ at these points and an edge (v, v') of the blending that crosses the glueing locus, with v in P and v' in Q. Consider an hyperplane H whose intersection with P is v'_P and perform a projective transformation which sends H at infinity. Then P gets an end, and, up to an affine transformation, we can assume this end to be the product of a simplex S by a vertical interval $[0; +\infty[$. Assume furthermore that v is the vertex of P which is sent to the highest place (this always can be done by a new affine trasformation that does not change the end). Finally truncate the polytope at heigth $\dfrac{1}{2}$.

Perform the same transformation for Q, taking $S\times]-\infty; 1[$ as end and sending v' to the lowest place, truncating Q at the same height as P. The polytope obtained by glueing the two pieces is $P\#Q$. Now, from this polytope, let the Q-piece fall on the P-piece. The first two vertices that will met are v and v'. The polytope we get just after their meeting is obtained by a flip along the desired edge. Hence the result. □
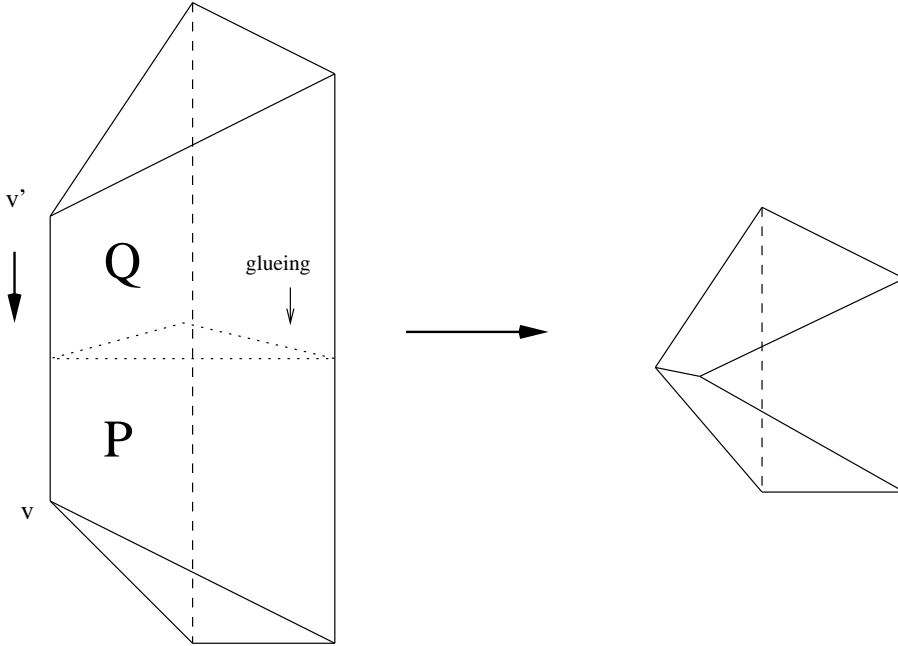

A flip has been performed

PROPOSITION **1.5** *The hypothesis is true for 3-dimensional polytopes.*

PROOF This results from Steinitz's caracterization of graphs of 3-polytopes, namely:

<u>Theorem (Steinitz,1922)</u> A graph is the 1-skeleton of a 3-dimensional polytope if and only if it is planar and 3-connected (and has ≥ 4 vertices). $\diamond$

Obviously, the polytope is simple if and only if the graph is trivalent. Consider the graph G of a 3-dimensional polytope P and an edge e of G. Perform now the combinatorial flip of P along e and call G' its graph. This can be done except in the case, that we will drop, where the edge belongs to a triangular facet of G, in which case its extremal facets are not disjoint. Thus, we have to show that G' is the graph of some 3-polytope if e has disjoint extremal facets.

First, we can claim that the new graph G' is planar thanks to the following picture:
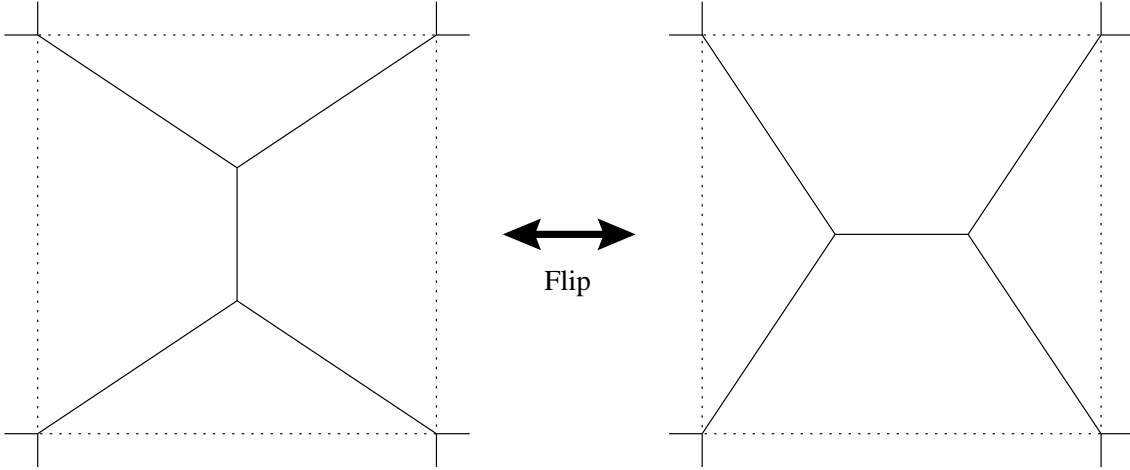
The flip occurs inside the box. The resulting graph is planar.

If G' is not 3-connected, consider a pair $\{v_0, v_1\}$ that disconnects him. If neither v_0 nor v_1 were one of the two new vertices, then G itself would be disconnected by this same pair. If v_0 and v_1 were the two new vertices, G would be disconnected by the vertices that have disappeared. Hence we can consider that v_0 is a vertex of G and v_1 is new. We can then maintain that the triple T formed by v_0 and the two extremities of e disconnects G. Now, there is vertex w that is adjacent to v_0 and cannot be joined in $G\backslash T$ to any other such vertex. Consider finally the two facets of P that contains v_0 and w. Each one contains one of the two extremities of e else this would contradict what preceeds and none contains both as their intersection is the edge $[v_0; w]$. So they are the extremal facets of e and are not disjoint. The proposition is proved. $\square$

1.3   Wedges

Let P be a simple polytope with nonempty interior in $\mathbb{R}^n$, F a facet of P, and l an affine function on $\mathbb{R}^n$ which is nonnegative on P but vanishes on F. We construct an other polytope $W(P, N)$ by intersecting, in $\mathbb{R}^n \times \mathbb{R}$, the $P \times \mathbb{R}_+$ with the half-space $\{(X, z) \in \mathbb{R}^n \times \mathbb{R}, z \leq l(X)\}$. This polytope is called the wedge over P with foot F.

We generalise this notion by defining, for $k \geq 1$:
$W^k(P, F) = \{(X, z_1, ..., z_k) \in P \times (\mathbb{R}_+)^k, z_1 \leq z_2 \leq ... \leq z_k \leq l(X)\}$.

DEFINITION **1.8** *The facets that contain F will be called the* large *facets of $W^k(P, F)$, the others well be called the* small *ones. The large ones will be noted $F_i, 0 \leq i \leq k$ and the small ones will be noted like their projections on P.*

PROPOSITION **1.6** *A vertex of $W^k(P, F)$ is either a vertex of F or belongs to all but one large facet. Moreover, for any vertex v in $P\backslash F$ and large facet F_i, there is a vertex of $W^k(P, F)$ that projects on v and does not belong to F_i.*

A vertex of F will be noted by the small facets that contain it, an other vertex will be noted by the small facets that contain it, followed by a hat on the large facet that does not contain it. For example, consider for P a square, $1, 2, 1', 2'$ its edges. Then, in $W^3(P, 1)$ the vertex (2) corresponds to the vertex $(1, 2)$ of P and the vertex $(1'2\hat{1}_2)$ corresponds to the one which lies on the facets 1_0, 1_1, 1_3, $1'$ and 2.

COROLLARY **1.7** *(i) The polytope $W^k(P, F)$ has dimension $d_P + k$, has $n_P + k$ facets, among which $k + 1$ are large, and has $n'_F + k(n'_P - n'_F)$ vertices;*
(ii) the automorphism group of $W^k(P, F)$ acts transitively on the large facets;
(iii) the wedge over $W^k(P, F)$ with foot a large facet is isomorphic to $W^{k+1}(P, F)$.

The proofs of the proposition an the corollary are left to the reader.

1.4   The polytope $Q4$

There is exactly one combinatorial type of 4-polytope with 9 vertices whose diameter is equal to 5. This polytope is called $Q4$ and can be considered as the "smallest nontrivial sharp polytope". It has first been discovered by Klee and Walkup [K-W]. Here is one of its constructions:

Consider a 4-dimensional regular hypercube and choose some point that we consider as the "north pole". The facets containing it are called the upper facets, and we note them $1, 2, 3, 4$ whereas their respective opposites, called the lower facets, are noted $1', 2', 3', 4'$. Truncate now this hypercube just under the equator. This introduces a

ninth facet, that we note N. Finally perform four flips of edges along the orbit of the edge $(1234')$—$(1'234')$ under the natural action of $\mathbb{Z}_4$. Now, $Q4$ is the resulting polytope. In this polytope, the minimal length of a path joining the vertices $(12'34')$ and $(1'23'4)$ is 5.

N.B.: I like this construction because it illustrates the strength of flipping (and especially edge flipping) in suppressing "small" paths between two points.

Note the following (easy to verify) property of the polytope $Q4$:

REMARK **1.8** *Any nonrevisiting path from* $(12'34')$ *to* $(1'23'4)$ *must meet (at least) one of the following vertices:*

$(2'34'N), (1'2'3N), (1'2'4N), (1'3'4N), (12'4'N), (13'4'N), (23'4'N), (1'23'N)$.

2   Eliminating nonrevisiting paths

2.1   Path-Flips

We describe here an efficient way to suppress nonrevisiting paths (as long as the hypothesis can be trusted). The idea behind this is that given a nonrevisiting path, it is possible to perform a transformation, that we will call path-flip, of the polytope, so that one of its vertices now lies on a facet it has already visited. Hence this path becomes revisiting.

Consider a polytope P and a nonrevisiting path p in P such that the start facet of p is disjoint from any of its arrival facets. Then we can get a polytope that we note $\tilde{P}_p$ in the following manner:

Assume $p = (v_0, ..., v_l)$. Then the edge (v_0, v_1) is flippable in P as its extremal facets, which are the start facet and first arrival facet of p, are disjoint. In the flipped polytope $\tilde{P}_1$, v_2 is adjacent to exactly one new vertex v_1'. Consider in $\tilde{P}_1$ the path $p_1 = (v_1', v_2, ..., v_l)$. It has the same start facet as p and its arrival facets are the same (except the first of p). As the flip of an edge only connects the extremal facets of that edge, the path p_1 in $\tilde{P}_1$ satisfies the same hypothesis as p in P. Hence the preceding construction can be repeated. This gives finally, after l flips, the polytope $\tilde{P}_p$.

We say that $\tilde{P}_p$ is obtained from P by a path-flip along p.

Assume now that we have several disjoint nonrevisiting paths $p_1, ..., p_k$ in P such that all of them satisfy the formentioned property, no two of them have the same start facet and none of them has for start facet an arrival facet of anoter of these paths.

Then we can perform the path-flips altogether, which yields a polytope $\tilde{P}_{p_1,...,p_k}$.

To see this by induction, we only have to verify that the strict transforms of $p_2, ..., p_k$ satisfy the same hypothesis in $\tilde{P}_{p_1}$. This is clear as their start and arrival facets

have not changed and besides, their start facets cannot having been connected with another facet since they are neither the start facet not an arrival facet of p_1. So $\tilde{P}_{p_1,...,p_k}$ exists and is said obtained from P by path-flips along the paths $p_1,...,p_k$.

2.2 Vertices without nonrevisiting path

Here comes our principal result:

PROPOSITION **2.1** *Consider two polytopes P and Q of the same dimension, points v_P, v'_P of P and v_Q, v'_Q of Q, a one-to-one correspondance ϕ between the facets of P containg v'_P and the facets of Q containg v'_Q, and a family of paths p_i in Q such that:*

- *(i) every p_i has v'_Q as origin;*

- *(ii) no p_i contains v_Q;*

- *(iii) for $i \neq j$, the intersection of p_i and p_j, is reduced to v'_Q;*

- *(iv) every p_i in nonrevisiting;*

- *(v) the virtual start facets of the p_i are pairwise different;*

- *(vi) in P, any nonrevisiting path from v_P to any vertex adjacent to v'_P meets every virtual start facet of the p_i;*

- *(vii) in Q, let v be any vertex adjacent to v'_Q. Then, if there exists a nonrevisiting path from v to v_Q which doesn't meet any p_i, then it meets the start facet of (v'_Q, v) and v_P lies on the image by ϕ of this facet;*

- *(viii) for any i, v_Q lies on the start facet of p_i and v_P on its image by ϕ.*

Then, from $P\#_\phi Q$, we can perform path-flips along the paths p_i whose origin has been replaced by their virtual origin, and, in the resulting polytope $\widetilde{P\#_\phi Q}_{p_i}$, there is no nonrevisiting path from v_P to v_Q.

PROOF Let's begin by showing that the path-flips can be performed. We call p'_i the path lying in $P\#Q$ where the origin of p_i has been replaced by its vitual origin. For any i, the start facet of p'_i is a stable left facet and all its arrival facets are stable right facets. Hence its start facet is disjoint from all its arrival facets. The point (iv) shows that p'_i is nonrevisiting. The paths p'_i are pairwise disjoint by (iii). The start facet of p'_i is the left extremal facet of the edge containing its origin. They are pairwise different by point (v). Also, every start facet of a p'_i is a stable left facet, then cannot be any arrival facet of a p'_j since all of them are stable right facets.

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The concomitance of all these properties allows us to claim that the path-flip along all the paths p_i' can be constructed, which yields the polytope noted $\widetilde{P\#_\phi Q}_{p_i}$. Notice also that v_Q is actually one of its vertex thanks to point (ii), and that v_P is obviously one also.

We now prove that every path in $\widetilde{P\#_\phi Q}_{p_i}$ whose origin is v_P and end is v_Q is revisiting. Consider such a path p and call v its first vertex which is not a left vertex. Then:

<u>First case:</u> The vertex v is a new vertex. Then, it comes from the flip of some path p_i. Clearly, v does not lie on the strict transform by the path-flip of the facet of $P\#_\phi Q$ corresponding to the glueing of the start facet of p_i and its image by ϕ. But by $(viii)$, both v_P and v_Q lie on it. This facet is then revisited by p.

<u>Second case:</u> The vertex v is a right vertex. Then, the first part of p, from v_P to the vertex immediately before v is in P. Then, by (vi), either it is revisiting, and then p is, or it meets every virtual left facet, and has left all of them when arrived at v. Then either p is revisiting or it avoids all the new vertices as every such vertex lies on some virtual left facet. By (vii), we can claim that such a path meets the strart facet of (v_Q', v') where v' is the vertex immeditely following the last left vertex (v' might a priori differ from v as the path could "return into P" after having reached v), and even that it meets this facet after v'. As we also have assumed that v_P lies on this facet, we can maintain that p revisits it.

All in all, every path from v_P to v_Q in $\widetilde{P\#_\phi Q}_{p_i}$ is revisiting. $\square$

3   The counterexample

We just now have to find datas that satisfy the conditions of proposition 2.1. Here are some:

Consider as P the square of the wedge W^3Q4 and as Q the polytope $W^{10}Q4$. Let $v_P' = \begin{pmatrix} 1'23'4\hat{N}_0 \\ 1'23'4\hat{N}_0 \end{pmatrix}$, $v_Q' = (12'34'\hat{N}_0)$ and ϕ the following correspondance of facets:
$1'_{up,left} \leftrightarrow N_{1,right}, 2_{up,left} \leftrightarrow N_{2,right}, 3'_{up,left} \leftrightarrow N_{3,right}, 4_{up,left} \leftrightarrow N_{4,right}$,
$1'_{down,left} \leftrightarrow N_{5,right}, 2_{down,left} \leftrightarrow N_{6,right}, 3'_{down,left} \leftrightarrow N_{7,right}, 4_{down,left} \leftrightarrow N_{8,right}$,
$N_{1,up,left} \leftrightarrow 1_{right}, N_{2,up,left} \leftrightarrow 2'_{right}, N_{1,down,left} \leftrightarrow 3_{right}, N_{2,down,left} \leftrightarrow 4'_{right}$,
$N_{3,up,left} \leftrightarrow N_{9,right}, N_{3,down,left} \leftrightarrow N_{10,right}$.

Consider now the following paths in $W^{10}Q4$:
$p_1 = (12'34'\hat{N}_0) \to (12'34'\hat{N}_9) \to (2'34') \to (1'2'3) \to (1'2'4) \to (1'3'4)$ and
$p_2 = (12'34'\hat{N}_0) \to (12'34'\hat{N}_{10}) \to (12'4') \to (13'4') \to (23'4') \to (1'23')$.

Finally, let $v_P = \begin{pmatrix} 12'34'\hat{N}_1 \\ 12'34'\hat{N}_1 \end{pmatrix}$ and $v_Q = (1'23'4\hat{N}_0)$.

<u>Claim:</u> These datas satify the hypothesis of propostion 2.1, hence this proposition

applies, leading to a counterexample to the nonrevisitiong path conjecture.

This is easy to verify. The first three points are obvious. The point (iv) is an immediate verification. The virtual start facet of p_1 is $N_{0,up}$, the one of p_2 is $N_{0,down}$, hence (v). Besides, both contain v_P, hence (vi).

Consider a vertex v of Q which is adjacent to v'_Q. If v has the form $(12'34'\hat{N}_k)$, then every nonrevisiting path from v to v_Q meets some p_i according to remark 1.8. If v is on N, it is $(12'4')$ (on p_2) or $(2'34')$ (on p_1). The last two possibilities are $v = (11'34'\hat{N}_0)$ or $v = (12'33'\hat{N}_0)$. Consider in this case a nonrevisiting path p from v to v_Q that avoids p_1 and p_2. The path $p' = (v'_Q, p)$ must be revisiting, else this would contradict remark 1.8. The only facet that p' can revisit is its start facet and so p meets it. Besides, this facet is $2'$ and its image by ϕ is $N_{2,up}$ if $v = (11'34'\hat{N}_0)$ and this facet is $4'$ and its image by ϕ is $N_{2,down}$ if $v = (12'33'\hat{N}_0)$. In both cases, v_P lies on this facet. So (vii) is verified.

The start facet of p_1 is N_9, its image by ϕ is $N_{3,up}$; the start facet of p_2 is N_{10}, its image by ϕ is $N_{3,down}$. Hence $(viii)$. The claim is proved.

REMARK **3.1** *As v_P and v_Q belong to two common facets, the intersection of these two facets is a 12-dimensional polytope without nonrevisiting path from v_P to v_Q.*

Indeed, it is possible that the analyse of the trasformation on this polytope allows us to decrease even more the dimension or number of facets of a counterexample.

Another fact is that it is sometimes believed that the sharpest polytopes are neighbourly dual (remark that $Q4$ is such a polytope). The ones we have described are not even 2-neighbourly dual, so it might be possible, following this way, to improve the counterexample.

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Bosio Frédéric
Université de Poitiers
UFR Sciences SP2MI
Département de Mathématiques
UMR CNRS 6086
Teleport 2
Boulevard Marie et Pierre Curie
BP 30179
86962 Futuroscope Chasseneuil CEDEX

e-mail : bosio@math.univ-poitiers.fr